Last time:

Derivatives:

$$\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

Integrals:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Fund. Thm of Calculus (in vector setting)

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

$$\vec{R}\,'(t) = \vec{r}(t)$$

We denote $\vec{R}(t) = \int \vec{r}(t)\,dt$ (indefinite integral)

Ex: $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

$$\int \vec{r}(t)\,dt = 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + \vec{C} = \vec{R}(t)$$

$$\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt = 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k}\Big|_0^{\frac{\pi}{2}} = 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k} = \vec{R}\left(\tfrac{\pi}{2}\right) - \vec{R}(0)$$

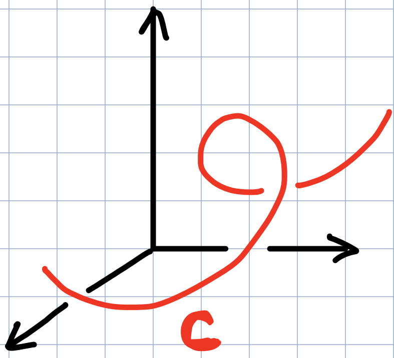
$$\vec{R}\left(\tfrac{\pi}{2}\right) = \left\langle 2\sin\tfrac{\pi}{2}, -\cos\tfrac{\pi}{2}, \left(\tfrac{\pi}{2}\right)^2 \right\rangle = \left\langle 2, 0, \tfrac{\pi^2}{4} \right\rangle; \quad \vec{R}(0) = \langle 2\sin(0), -\cos(0), 0^2 \rangle = \langle 0, -1, 0 \rangle$$

Today: Arc length,

[no curvature],

TNB Frame

Arc length



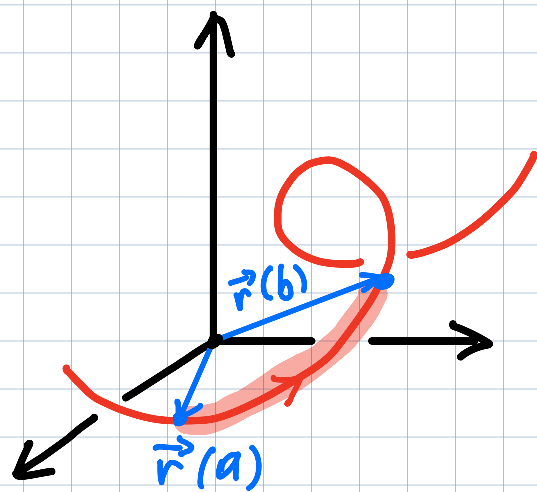
curve C with vector equation

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \qquad a \le t \le b$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

length of the curve

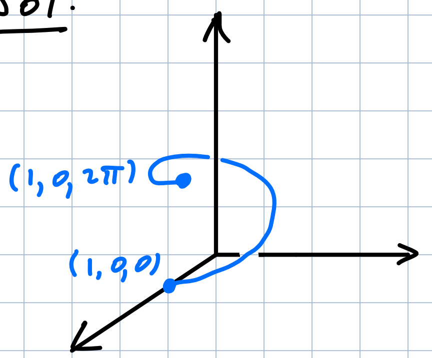
$$= \int_a^b |\vec{r}\,'(t)|\, dt$$



Ex: $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$

Find the arc length between $(1,0,0)$ and $(1,0,2\pi)$

Sol:



1) $\langle 1,0,0\rangle = \vec{r}(0)$, $\quad \langle 1,0,2\pi\rangle = \vec{r}(2\pi)$

2) $\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$

$\Rightarrow |\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$

$\Rightarrow L = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi$

- arc length function: $s(t) = \int_a^t |\vec{r}\,'(u)|\,du$ - length of the piece of the curve between $\vec{r}(a)$ and $\vec{r}(t)$

Ex: $\boxed{\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}}$, $a = 0$. Find the arc length function.

Sol: $s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$

$\Rightarrow t = \frac{s}{\sqrt{2}}$

One can express $\boxed{}$ via s: $\vec{r} = \cos\frac{s}{\sqrt{2}}\,\vec{i} + \sin\frac{s}{\sqrt{2}}\,\vec{j} + \frac{s}{\sqrt{2}}\,\vec{k}$ $\Big\}$ param. of the curve via arc length

At a point $\vec{r}(t)$ on C:

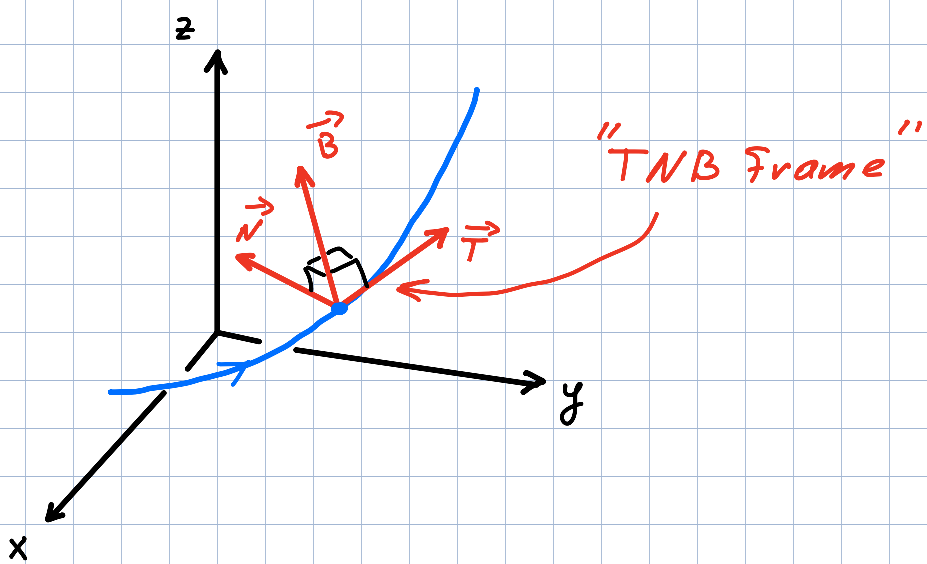
unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

(principal) unit normal vector $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ ← points in the direct. in which the curve is turning

Rmk: $\vec{T}(t) \cdot \vec{T}'(t) = 0$ (since $|\vec{T}(t)| = 1$ for all t)

binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

$\{\vec{T}(t), \vec{N}(t), \vec{B}(t)\}$ – "TNB frame" – 3 orthogonal vectors



Ex: $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$. Find $\vec{T}, \vec{N}, \vec{B}$

Sol:

$$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}, \quad |\vec{r}'(t)| = \sqrt{2}$$

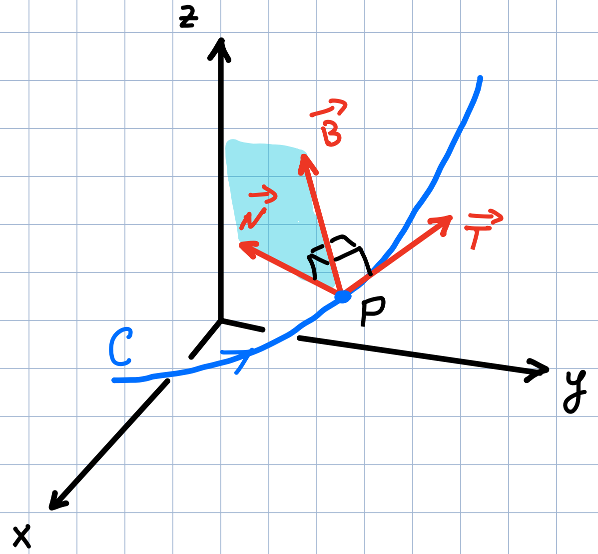
$$\vec{T}(t) = \frac{1}{\sqrt{2}}\left(-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}\right)$$

$$\Rightarrow \vec{T}'(t) = \frac{1}{\sqrt{2}}\left(-\cos t\,\vec{i} - \sin t\,\vec{j}\right)$$

and $|\vec{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{(-\cos t)^2 + (-\sin t)^2} = \frac{1}{\sqrt{2}}$

$$\vec{N}(t) = \sqrt{2}\,\vec{T}'(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\sin t\,\vec{i} - \cos t\,\vec{j} + \vec{k}\right)$$



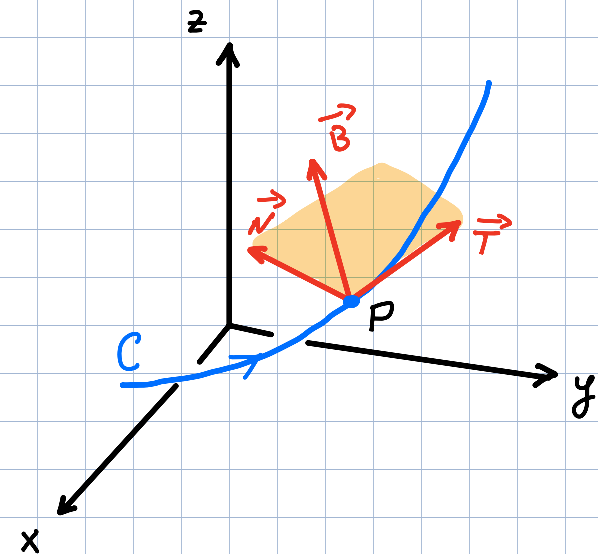
For a point P on C,

plane through $\vec{N}$ and $\vec{B}$

normal plane of C at P

plane through $\vec{T}, \vec{N}$

osculating plane of C at P



Rmk: For a plane curve C, it is the plane containing C.

Rmk: $\vec{r}''(t)$ is in the osculating plane

Ex: Find equations of normal and osculating planes for $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ at $P(0, 1, \frac{\pi}{2})$

Sol: $P(0,1,\frac{\pi}{2})$ corresponds to $t=\frac{\pi}{2}$

1) normal plane – passes through P,
has normal vector $\vec{T}(\frac{\pi}{2})$
(or just $\vec{r}'(\frac{\pi}{2})$)

$$\left.\begin{array}{l} P(0,1,\frac{\pi}{2}) \\ \vec{n} = \langle -1, 0, 1\rangle \end{array}\right\} \Rightarrow \quad -1(x-0) + 0(y-1) + 1(z-\tfrac{\pi}{2}) = 0$$

or $\quad -x + z - \frac{\pi}{2} = 0$

2) osculating plane – passes through P,
has normal vector $\vec{B}(\frac{\pi}{2})$

$$\left.\begin{array}{l} P(0,1,\frac{\pi}{2}) \\ \vec{n} = \langle 1, 0, 1\rangle \end{array}\right\} \Rightarrow \quad 1(x-0) + 0(y-1) + 1(z-\tfrac{\pi}{2}) = 0$$

or $\quad x + z - \frac{\pi}{2} = 0$

Rmk: $\vec{r}''(t)$ is in the osculating plane

Ex: $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$

Find $\vec{T}, \vec{N}, \vec{B}$ at $P(1, \frac{2}{3}, 1)$

Sol: $(1, \frac{2}{3}, 1)$ corresp. to $t = 1$

$\vec{r}\,'(t) = \langle 2t, 2t^2, 1 \rangle$ $\qquad$ $\vec{r}\,''(t) = \langle 2, 4t, 0 \rangle$

$\vec{r}\,'(1) = \langle 2, 2, 1 \rangle$ $\qquad$ $\vec{r}\,''(1) = \langle 2, 4, 0 \rangle$

$$\vec{T}(1) = \frac{\langle 2, 2, 1 \rangle}{\sqrt{4+4+1}} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

$\vec{b} = \vec{r}\,'(1) \times \vec{r}\,''(1)$ – normal to osculating plane

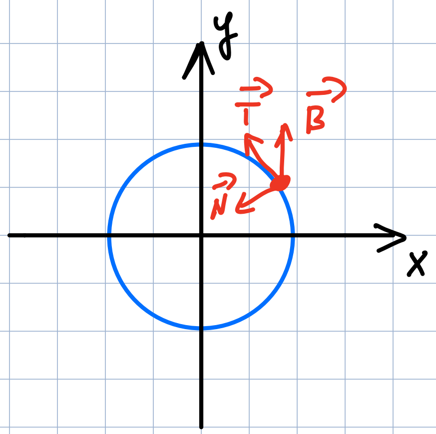
$$\vec{B}(1) = \frac{\vec{b}}{|\vec{b}|} = \frac{\langle -4, 2, 4 \rangle}{\sqrt{(-4)^2 + 2^2 + 4^2}} = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

$$\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\vec{i} + 2\vec{j} + (8-4)\vec{k} = \langle -4, 2, 4 \rangle$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{9} \begin{vmatrix} i & j & k \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{9}\left((1-4)i + (4-(-2))j + (-4-2)\vec{k} \right)$$

$$= \left\langle -\frac{1}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

Ex: $\vec{r}(t) = \langle 3\cos t + 1,\ 3\sin t,\ 0 \rangle$



$$\left.\begin{aligned} \vec{r}\,'(t) &= \langle -3\sin t,\ 3\cos t,\ 0 \rangle \\ |\vec{r}\,'(t)| &= 3 \end{aligned}\right\} \Rightarrow \vec{T}(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$\left.\begin{aligned} \vec{T}\,'(t) &= \langle -\cos t, -\sin t, 0 \rangle \\ |\vec{T}\,'(t)| &= 1 \end{aligned}\right\} \Rightarrow \vec{N}(t) = \vec{T}\,'(t)$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \vec{k}$$